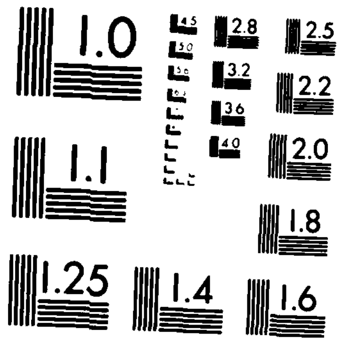
AD-A147 360 A COUNTEREXAMPLE TO A RESULT OF STRANG AND FIX CONCERNING CONTROLLED APPROXIMATION(U) WISCONSIN UNIV-MADISON MATHEMATICS RESEARCH CENTER R Q JIA 1/1

UNCLASSIFIED SEP 84 MRC-TSR-2743 DAAG29-80-C-0041 F/G 12/1 NL

END



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

2

AD-A147 360

MRC Technical Summary Report # 2743

A COUNTEREXAMPLE TO A RESULT OF STRANG AND FIX CONCERNING CONTROLLED APPROXIMATION

Rong-Qing Jia

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53705

September, 1984

(Received June 4, 1984)

Approved for public release
Distribution unlimited

Sponsored by

U.S. Army Research Office
P.O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, D.C. 20550

UNIVERSITY OF WISCONSIN-MADISON
MATHEMATICS RESEARCH CENTER

A COUNTEREXAMPLE TO A RESULT OF STRANG AND FIX CONCERNING CONTROLLED APPROXIMATION

Rong-Qing Jia

Technical Summary Report #2743
September 1984

ABSTRACT

A result of Strang and Fix states that if the order of controlled approximation from a collection of locally supported elements is k, then there is a finite linear combination Ω of those elements and their translates such that any polynomial of degree less than k can be reproduced by Ω and its translates. This paper gives a counterexample to their result.

AMS (MOS) Subject Classifications: 41A25, 41A15, 41A63, 65N30

Key Words: Controlled Approximation, Approximation Order, Box Splines, Finite Element Analysis, Fourier Series

Work Unit Number 3 - Numerical Analysis and Scientific Computing

---

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041.
This material is based upon work supported by the National Science Foundation under Grant No. MCS-8210950.

SIGNIFICANCE AND EXPLANATION

In finite element analysis it is important to choose appropriate trial functions. One approach to such a choice has been described by Strang as follows: Choose one or more locally supported functions $\phi_1, \ldots, \phi_n$, rescale the independent variable, and translate the functions just constructed. A principal question in finite element analysis is how well a given function can then be approximated by those trial functions. This is measured by the order of approximation. Sometimes it is required that the coefficients in the process of approximation be bounded. Such an approximation is said to be controlled. A result of Strang and Fix states that if a collection of locally supported elements has controlled approximation of order k, then there is a finite linear combination ψ of those elements and their translates such that any polynomial of degree less than k can be reproduced by ψ and its translates. This result has been questioned for some time. Recently, the rapid development of the theory of multivariate spline functions has given a new impetus to the controlled approximation problem. It becomes significant to answer the question whether their result is correct. This paper gives a counterexample to the "result" of Strang and Fix. ~~Our~~ conclusion suggests that we should redefine the controlled approximation order should be refined. Hopefully, the result in this paper will lead to further development of finite element analysis.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

A COUNTEREXAMPLE TO A RESULT OF STRANG AND FIX
CONCERNING CONTROLLED APPROXIMATION

Rong-Qing Jia

This paper concerns the so-called controlled approximation. This concept was introduced by [S] in 1970. In that paper, Strang described a systematic approach to the choice of trial functions used in the finite element analysis (Also see [FS] for some related results). To describe his approach, we first introduce some notation. As usual, we mean by $W^{k,p}(\mathbb{R}^n)$ the usual Sobolev space with the norm

$$\|u\|_{k,p} := \sum_{|\alpha|\leq k} \|D^\alpha u\|_{L^p} .$$

(Here we use the standard multi-index notation). The seminorm $|\cdot|_{k,p}$ on $W^{k,p}(\mathbb{R}^n)$ is defined to be

$$|u|_{k,p} := \sum_{|\alpha|=k} \|D^\alpha u\|_{L^p} .$$

When $k = 0$, $|\cdot|_{k,p}$ is a norm, and we write

$$\|u\|_p := |u|_{0,p} = \|u\|_{L^p} .$$

By $W_c^{k,p}(\mathbb{R}^n)$ we denote the space of all functions in $W^{k,p}(\mathbb{R}^n)$ that have compact support. By $\ell^p(\mathbb{Z}^n)$ we mean the space of the mappings w from $\mathbb{Z}^n$ to $\mathbb{R}$ which satisfy

$$\|w\|_p := \left(\sum_j |w(j)|^p\right)^{1/p} < \infty .$$

Strang's approach can be described as follows: Choose one or more functions $\phi_1,\ldots,\phi_N$ in $W_c^{k,2}(R^n)$, rescale the independent variable, obtaining

---

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041.
This material is based upon work supported by the National Science Foundation under Grant No. MCS-8210950.

$h^{-n/2}\phi_i(\cdot/h)$ with $h > 0$ $(i = 1,\ldots,N)$, and translate the functions just constructed, replacing x by $x - jh$. Thus we have a family of trial functions:

$$\phi^h_{i,j}(x) := h^{-n/2}\phi_i\left(\frac{x}{h} - j\right), \quad i = 1,\ldots,N, \quad j \in \mathbb{Z}^n .$$

When $h = 1$, we omit the superscript h in $\phi^h_{i,j}$. Note that the coefficient $h^{-n/2}$ normalizes $\phi^h_{i,j}$ in the sense that

$$\|\phi^h_{i,j}\|_2 = \|\phi_i\|_2 .$$

A principal question in finite element analysis is the degree of approximation which can be achieved by the span of $\phi^h_{i,j}$. Concerning this problem, Strang and Fix state the following result (see [SF; Theorem II]):

<u>Theorem A</u>. Suppose $\phi_1, \ldots, \phi_n$ are in $W^{k,2}_c(\mathbb{R}^n)$. Then the following conditions are equivalent:

$(i)_k$ *for each* u *in* $W^{k+1,2}$ and $h > 0$, there are weights $w^h_{i,j}$ such that

$$(1) \qquad |u - \sum_{i,j} w^h_{i,j}\phi^h_{i,j}|_{s,2} \le \text{const}_s\, h^{k+1-s}|u|_{k+1,2} \quad \text{for} \quad s \le k ,$$

$$(2) \qquad \|w^h_i\|_2 \le \text{const}\, \|u\|_2 .$$

(Here w^h_i is the mapping $j \to w^h_{i,j}$, $j \in \mathbb{Z}^n$.)

$(ii)_k$ there is a <u>finite</u> linear combination Ω of $\phi_1,\ldots,\phi_n$ and their translates $\phi_{i,j}$ such that

for $|\alpha| \le k$, $\sum_{j\in\mathbb{Z}^n} j^\alpha\Omega(x - j)$ is a polynomial in $x_1,\ldots,x_n$ with leading term x^α.

If a collection $\Phi = \{\phi_1, \ldots, \phi_N\}$ satisfies $(i)_{k-1}$ but not $(i)_k$, then we say that Φ has controlled approximation order k. Note that (2) constrains the coefficients $w^h_{i,j}$. This is the reason why [S] calls such approximation <u>controlled</u>.

The implication of $(i)_k$ from $(ii)_k$ is in no doubt. This has been proved by several authors, in particular, by [SF] (see the references cited by [S] and [FS_2]). But, the other direction, i.e., the implication of $(ii)_k$ from $(i)_k$ has been proved <u>only</u> for $N = 1$ by [SF]; the proof for the case $N > 1$ is not discussed in [SF].

Recently, the rapid development of the theory of multivariate spline functions has given a new impetus to the controlled approximation problem. In [DM], Theorem A was cited to determine the controlled approximation order from certain bivariate box splines. In order to advance the theory of controlled approximation, it becomes important to answer the question whether $(i)_k$ implies $(ii)_k$.

As it turns out, $(i)_k$ does <u>not</u> imply $(ii)_k$ in general. We shall give a counterexample in the following. In our example, $n = 2$, $N = 4$, and $\phi_1, \ldots, \phi_4$ are splines with compact support, which are constructed from box splines. Thus we have to recall box splines. Before doing so, we need to introduce more notation. The vectors e_i $(i = 1,2,3)$ in $\mathbb{R}^2$ are given by

$$e_1 = (1,0), \quad e_2 = (0,1) \quad \text{and} \quad e_3 = (1,1) .$$

The difference operators ∇_i are given by the rule

$$\nabla_i f = f - f(\cdot - e_i), \quad \text{for} \quad f : \mathbb{Z}^2 \to \mathbb{R} .$$

The differential operators D_i are given by

$$D_i = \frac{\partial}{\partial x_i} \; (i = 1,2) \quad \text{and} \quad D_3 = D_1 + D_2 .$$

Moreover, we denote by π_k the space of all polynomials in two variables of total degree no more than k. Let us recall the definition of box splines from [BH]. For a sequence $\Xi = (\xi_i)_1^n$ of vectors in $\mathbb{R}^n$, the box spline M_Ξ is defined to be the distribution given by

$$\langle M_\Xi, \phi\rangle = \int_{[0,1]^n} \phi\Big(\sum_{i=1}^{n} \lambda(i)\xi_i\Big)d\lambda$$

for any C^∞-locally supported function ϕ. If Ξ consists of e_1, e_2 and e_3, repeated r, s and t times, respectively, then we write $M_{r,s,t}$ instead of M_Ξ. In particular,

$$M_{r,s,0}(x_1,x_2) = M_r(x_1)M_s(x_2) .$$

Here M_r is the usual (univariate) B-spline of order r at the knot sequence $0,1,\ldots,r$:

$$M_r(x) = [0,1,\ldots,r](\cdot - x)_+^{r-1} ,$$

where $[\rho_0,\rho_1,\ldots\rho_r]f$ means the divided difference of f at points $\rho_0,\rho_1,\ldots,\rho_r$.

Now we can formulate our collection $\Phi = \{\phi_1,\phi_2,\phi_3,\phi_4\}$. Our candidates are $M_{2,0,1}$, $M_{2,1,0}$, $M_{0,2,1}$ and $M_{1,2,0}$. But we want to modify them to suit our purpose. First, for the technical reason which will be clear later, we want to shift those splines. Thus we define the shift operator τ by

$$\tau f := f(\cdot + e_3) .$$

Second, more important, those splines are not in the space $W^{1,2}$. So we have to smooth them. This can be done by convolving them with a sufficiently smooth function. We should be careful in doing so, however, because we want to keep Φ from satisfying $(ii)_1$. Thus we introduce the convolution operator σ as follows:

$$\sigma f := \int f(x_1 - t, x_2 - t)2M_1(2t + \tfrac{1}{2})dt$$

$$= 2\int_{-1/4}^{1/4} f(x_1 - t, x_2 - t)dt .$$

Note that σ and τ commute with each other. Let

$$\phi_1 := \sigma\tau\left(M_{2,0,1}\right), \qquad \phi_2 := \sigma\tau\left(M_{2,1,0}\right)$$

$$\phi_3 := \sigma\tau\left(M_{0,2,1}\right), \qquad \phi_4 := \sigma\tau\left(M_{1,2,0}\right).$$

Obviously, $\phi_\ell(\ell = 1,\ldots,4)$ are in $W^{1,2}$. Our result can be stated in the following

<u>Theorem</u>. <u>The collection</u> Φ <u>satisfies</u> $(i)_1$ <u>but not</u> $(ii)_1$.

Proof. To prove the first assertion, we start with the box spline $M_{1,1,1}$. This spline is well-known to finite element analysts as the standard linear element, which was first introduced by Courant [C]. The spline $M_{1,1,1}$ has the hexagon

$$\left\{\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3;\ 0 < \lambda_i < 1\right\}$$

as its support, and $M_{1,1,1}(1,1) = 1$. Let

$$\phi = \sigma\tau(M_{1,1,1}).$$

We claim

$$\sum_{j\in\mathbb{Z}^2} p(j)\phi(\cdot - j) = p \quad \text{for any } p \in \pi_1. \tag{3}$$

To this end, we recall the following elementary properties of box splines (see [BH]):

$$\Sigma M_\Xi(\cdot - j) = 1.$$

and

$$D_i\left(\Sigma a(j)M_\Xi(\cdot - j)\right) = \Sigma(\nabla_i a)(j)M_{\Xi\setminus e_i}(\cdot - j) \quad \text{if} \quad e_i \in \Xi. \tag{4}$$

For $p \in \pi_1$, $D_i p = \nabla_i p (i = 1,2)$ are constants, hence

$$D_i\left(\Sigma p(j)M_{1,1,1}(\cdot - j + e_3)\right) = \Sigma(\nabla_i p)(j)M_{\{e_1,e_2,e_3\}\setminus e_i}(\cdot - j + e_3)) = D_i p.$$

Moreover,

$$\Sigma p(j)M_{1,1,1}(0 - j + e_3) = p(0) .$$

This shows that

$$\Sigma p(j)(\tau M_{1,1,1})(\cdot - j) = p \quad \text{for} \quad p \in \pi_1 .$$

Let the convolution operator σ act on both sides of this equality. We have

$$\Sigma p(j)(\sigma\tau M_{1,1,1})(\cdot - j) = \sigma p .$$

It is easily seen that $\sigma p = p$ for any $p \in \pi_1$. This proves our claim (3). From the foregoing argument we also have

$$\Sigma(D_i\phi)(\cdot - j) = 1 .$$

Thus, by a standard argument (see the proof of [BH; Theorem 6]), we have, for $u \in W^{2,2}(\mathbb{R}^2) \cap C^2(\mathbb{R}^2)$ and $h > 0$,

$$(5) \qquad |u - \Sigma u(jh)\phi(\tfrac{\cdot}{h} - j)|_{s,2} \leq \text{const}\ h^{2-s}|u|_{2,2}, \quad s = 0,1 .$$

For simplicity, we write

$$w(j) = hu(jh)$$
$$v = \Sigma w(j)\phi_j^h$$

with

$$\phi_j^h = h^{-1}\phi(\tfrac{\cdot}{h} - j) .$$

Since $u \in L^2$, $w \in \ell^2$ for sufficiently small $h > 0$, and

$$\|w\|_2 \leq \text{const}\ \|u\|_2 .$$

We want to express v as an <u>infinite</u> linear combination of $\phi_{i,j}^h$. This is possible because of (4). Suppose $\gamma : \mathbb{Z}^2 \to \mathbb{R}$ satisfies $\nabla_1\gamma = w$. Then it follows from (4) that

$$\begin{aligned}\sum w(j)M_{1,1,1}(\cdot - j) &= \sum (\nabla_1\gamma)(j)M_{1,1,1}(\cdot - j) \\ &= D_1\Big(\sum \gamma(j)M_{2,1,1}(\cdot - j)\Big) = (D_3 - D_2)\Big(\sum \gamma(j)M_1(\cdot - j)\Big) \\ &= \sum (\nabla_3\gamma)(j)M_{2,1,0}(\cdot - j) - \sum (\nabla_2\gamma)(j)M_{2,0,1}(\cdot - j) .\end{aligned}$$

Recall that

$$\phi = \sigma\tau(M_{1,1,1}), \quad \phi_1 = \sigma\tau(M_{2,0,1}) \text{ and } \phi_2 = \sigma\tau(M_{2,1,0}).$$

We obtain

$$\sum w(j)\phi_j^h = \sum (\nabla_3\gamma)(j)\phi_{2,j}^h - \sum (\nabla_2\gamma)(j)\phi_{1,j}^h .$$

Thus v can be expressed as an infinite linear combination of $\phi_{1,j}^h$ and $\phi_{2,j}^h$. But we have trouble in keeping the coefficient sequence $\nabla_2\gamma$ bounded in L^2. To settle this trouble we shall use Fourier series. From now on, the letter i will be reserved for the imaginary unit $\sqrt{-1}$. For any $a \in \ell^2(\mathbb{L}^2)$, we denote by $\hat{a}$ the Fourier series given by

$$\hat{a}(\xi) := \Sigma\, a(j)e^{2\pi i j\cdot\xi} \quad , \quad \xi \in \mathbb{R}^2,$$

where $j\cdot\xi = j_1\xi_1 + j_2\xi_2$ is the inner product of j and ξ. We have the following elementary property:

$$(\nabla_k a)^{\wedge} = \Sigma\, \big(a(j) - a(j - e_k)\big)e^{2\pi i j\cdot\xi} = (1 - \exp(2\pi i\xi_k)\hat{a} \quad , \quad k = 1,2 . \tag{6}$$

Let

$$g_k(\xi) := 1 - \exp(2\pi i\xi_k), \quad k = 1,2 .$$

Then (6) tells us that

$$(\nabla_2 \hat{a}) = g_2\hat{a} = (g_2/g_1)(\nabla_1 a)^{\wedge} . \tag{7}$$

Let Q be the square $[0,1] \times [0,1]$. By Parseval's identity,

$$\|\hat{w}\|_{2,Q} := \left(\int_Q |\hat{w}|^2\right)^{1/2} = \|w\|_2 .$$

Thus if $\nabla_1\gamma = w$, then

$$\|\nabla_2\gamma\|_2 = \|(\nabla_2\gamma)^{\wedge}\|_{2,Q} = \|(g_2/g_1)(\nabla\gamma)^{\wedge}\|_{2,Q} = \|(g_2/g_1)\hat{w}\|_{2,Q} .$$

Since g_2/g_1 is unbounded, the norm $\|(g_2/g_1)\hat{w}\|_{2,Q}$ may be unbounded in general. Our scheme is to decompose w into a sum of two sequences a and

b. For a, we seek α such that $\nabla_1\alpha = a$ and $\|\nabla_2\alpha\|_2 \le \|w\|_2$, and for b, we find β such that $\nabla_2\beta = b$ and $\|\nabla_1\beta\|_2 \le \|w\|_2$. This decomposition of w can be done in the following way : Introduce the set

$$E := \{(\xi_1,\xi_2) \in Q; \ |g_1(\xi)| \ge |g_2(\xi)|\} ,$$

and its characteristic function

$$\chi_E(\xi) := \begin{matrix} 1 & \text{if} & \xi \in E \\ 0 & \text{if} & \xi \in Q\backslash E \end{matrix} .$$

Decompose $\hat{w}$ into $\hat{w}\chi_E + \hat{w}(1 - \chi_E)$. Obviously,

$$\|\hat{w}\chi_E\|_{2,Q} \le \|\hat{w}\|_{2,Q} \quad \text{and} \quad \|\hat{w}(1 - \chi_E)\|_{2,Q} \le \|\hat{w}\|_{2,Q} .$$

Hence we can expand $\hat{w}\chi_E$ and $\hat{w}(1 - \chi_E)$ in Fourier series :

$$\hat{w}\chi_E(\xi) = \Sigma\, a(j)e^{2\pi i j\cdot\xi} ,$$

$$\hat{w}(1 - \chi_E)(\xi) = \Sigma\, b(j)e^{2\pi i j\cdot\xi}$$

with

$$\|a\|_2, \|b\|_2 \le \|\hat{w}\|_{2,Q} = \|w\|_2 .$$

It follows that

$$\Sigma\, w(j)e^{2\pi i j\cdot\xi} = \hat{w}(\xi) = \Sigma\, \big(a(j) + b(j)\big)e^{2\pi i j\cdot\xi} .$$

By the uniqueness of Fourier series expansion, we have

$$w = a + b .$$

Since $\hat{a} = \hat{w}\chi_E$, and since $|g_2(\xi)| \le |g_1(\xi)|$ for $\xi \in E$, we have

$$|(g_2/g_1)\hat{a}(\xi)| \le |\hat{a}(\xi)| \quad \text{for} \quad \xi \in Q ,$$

and therefore we can find $c \in \ell^2(\mathbb{Z}^2)$ such that

(8) $$\hat{c} = (g_2/g_1)\hat{a} .$$

In addition,

$$\|c\|_2 = \|\hat{c}\|_{2,Q} \le \|\hat{a}\|_{2,Q} = \|a\|_2 \le \|w\|_2 .$$

Furthermore, (8) is equivalent to

$$g_1\hat{c} = g_2\hat{a} \quad , \quad \text{or} \quad (\nabla_1 c)^\wedge = (\nabla_2 a)^\wedge \quad ;$$

hence

$$\nabla_1 c = \nabla_2 a .$$

Having obtained a and c, we can find (uniquely) a sequence $\alpha : Z^2 \to R$ such that $\nabla_1\alpha = a$ and

$$(9) \qquad \alpha(0,j_2) = \begin{cases} \sum_{\ell=1}^{j_2} f(0,\ell) & \text{for } j_2 > 0 \\ 0 & \text{for } j_2 = 0 . \\ -\sum_{\ell=j_2+1}^{0} f(0,\ell) & \text{for } j_2 < 0 \end{cases}$$

Then

$$\nabla_1(\nabla_2\alpha) = \nabla_2(\nabla_1\alpha) = \nabla_2 a = \nabla_1 c .$$

But (9) tells us that

$$\nabla_2\alpha(0,j_2) = c(0,j_2) \quad \text{for all} \quad j_2 \in Z ;$$

therefore

$$\nabla_2\alpha = c .$$

Furthermore

$$\nabla_3\alpha = \alpha - \alpha(\cdot - e_3) = \alpha - \alpha(\cdot - e_2) + \alpha(\cdot - e_2) - \alpha(\cdot - e_3)$$

$$= \nabla_2\alpha + \nabla_1\alpha(\cdot - e_2) = \nabla_2\alpha + a(\cdot - e_2) ;$$

hence

$$\|\nabla_3\alpha\|_2 \le 2\,\|a\|_2 \le 2\,\|w\|_2 .$$

To sum up, we can find $\alpha : Z^2 \to R$ such that

$$\|\nabla_2\alpha\|_2 < \|w\|_2 , \quad \|\nabla_3\alpha\|_2 < 2\,\|w\|_2$$

and

$$\Sigma\, a(j)\phi_j^h = \Sigma\, (\nabla_3\alpha)(j)\phi_{2,j}^h - \Sigma\, (\nabla_2\alpha)(j)\phi_{1,j}^h \; .$$

Similarly, we can find $\beta : Z^2 \to R$ such that

$$\|\nabla_1\beta\|_2 < \|w\|_2 , \quad \|\nabla_3\beta\|_2 < 2\,\|w\|_2$$

and

$$\Sigma\, b(j)\phi_j^h = \Sigma\, (\nabla_3\beta)(j)\phi_{4,j}^h - \Sigma\, (\nabla_1\beta)(j)\phi_{3,j}^h \; .$$

Recall that

$$v = \Sigma\, w(j)\phi_j^h = \Sigma\, a(j)\phi_j^h + \Sigma\, b(j)\phi_j^h \; .$$

From (5) we observe that

$$|u - v|_{s,2} < \text{const}\; h^{2-s}|u|_{2,2} , \quad s = 0,1 \; .$$

Thus we have proved that the collection $\Phi = \{\phi_1,\phi_2,\phi_3,\phi_4\}$ satisfies $(i)_1$.

It remains to show that Φ does not satisfy $(ii)_1$. This will be done by contradiction. Suppose to the contrary that there is a finite linear combination Ω of $\phi_1, \ldots, \phi_4$ and their translates $\phi_{\ell,j}$ such that

for $|\alpha| < 1$, $\sum_{j\in Z^2} j^\alpha \Omega(x - j)$ is a polynomial in x_1, x_2 with leading term x^α.

Assume

$$\Omega = \sum_{\ell=1}^{4} \sum_{m\in I} c_{\ell,m}\phi_\ell(\cdot - m) \; , \tag{10}$$

where I is a finite subset of Z^2. We want to draw a contradiction from the above assumption. Let us first take a closer look to the function ϕ_1. By straightforward computation, we have

$$\phi_1(x_1,x_2) = M_2(x_1 - x_2 + 1)\psi(x_2)$$

with

$$\psi(x\ ) = \begin{cases} 2(x_2 + 5/4) & \text{for } -5/4 \le x_2 \le -3/4 \\ 1 & \text{for } -3/4 \le \ _2 \le -1/4 \\ -2(x_2 - 1/4) & \text{for } -1/4 \le x_2 \le -1/4 \\ 0 & \text{for } x < -4/5 \text{ or } x > 1/4 . \end{cases}$$

This motivates us to introduce the following functional F:

$$Fg := \lim_{\varepsilon \to +0} \left[D_3 g\left(\frac{1}{4} + \varepsilon, \frac{1}{4} + \varepsilon\right) - D_3 g\left(\frac{1}{4} - \varepsilon, \frac{1}{4} - \varepsilon\right)\right] \quad \text{for} \quad g \in W^{1,\infty}(\mathbb{R}^2) .$$

Just by computation, we obtain

$$F\big(\phi_1(\cdot - j)\big) = \begin{cases} 2 & \text{if } j = (0,0) \\ -2 & \text{if } j = (1,1) \\ 0 & \text{otherwise} . \end{cases}$$

It follows that

$$F\big(\Sigma\, j_2 \phi_1(\cdot - j)\big) = -2 .$$

By (4) we have

$$\Sigma\, j_2\, M_{2,1,0}(\cdot - j) = D_3 \Sigma \big(j_2(j_2 + 1)/2\big) M_{2,1,1}(\cdot - j)$$

$$= (D_1 + D_2)\ \Sigma \big(j_2(j_2 + 1)/2\big) M_{2,1,1}(\cdot - j) = \Sigma\, j_2 M_{2,0,1}(\cdot - j) ;$$

hence

(11) $$F\big(\Sigma\, j_2 \phi_2(\cdot - j)\big) = F\big(\Sigma\, j_2 \phi_1(\cdot - j)\big) = -2 .$$

By an argument similar to that used in proving (5), we observe that

$$\sum_j j_2 \phi_\ell(\cdot - j) \quad \text{is in} \quad \pi_1 \quad \text{for} \quad \ell = 3 \text{ or } 4 .$$

Therefore

(12) $$F\Big(\sum_j j_2 \phi_\ell(\cdot - j)\Big) = 0, \quad \ell = 3 \text{ or } 4 .$$

Now we consider the function

$$q := \Sigma\, j_2 \Omega(\cdot - j) .$$

On the one hand, $q \in \pi_1$; hence $F(q) = 0$. On the other hand, by (10),

$$q = \sum j_2\Omega(\cdot - j) = \sum_m \sum_{\ell=1}^{4} c_{\ell,m} j_2\phi_\ell(\cdot - m - j),$$

hence by (11) and (12),

$$F(q) = \sum_m \sum_{\ell=1}^{4} c_{\ell,m}\, F\Big(\sum j_2\phi_\ell(\cdot - m - j)\Big) = -2 \sum_{\ell=1}^{2} \sum_m c_{\ell,m}.$$

This shows that

$$\sum_{\ell=1}^{2} \sum_m c_{\ell,m} = 0 .$$

Similarly, one can show

$$\sum_{\ell=3}^{4} \sum_m c_{\ell,m} = 0 .$$

Finally, since $\Sigma\phi_\ell(\cdot - j) = 1$, we get

$$\sum \Omega(\cdot - j) = \sum_{\ell=1}^{4} \sum_m c_{\ell,m} \Big(\sum_j \phi_\ell(\cdot - m - j)\Big) = \sum_{\ell=1}^{4} \sum_m c_{\ell,m} = 0 .$$

This shows that Φ does not satisfy $(ii)_1$. Our proof is complete.

REFERENCES

[BH] C. de Boor and K. Höllig, B-splines from parallelepipeds, J d'Analyse, **42** (1983), 99-115.

[C] R. Courant, Variational method for the solution of problems of equilibrium and vibrations, Bull. Amer. Math. Soc. **49** (1943), 1-33.

[DM] W. Dahmen, C. A. Micchelli, On the approximation order from certain multivariate spline spaces, Journal of the Australian Mathematical Society, to appear.

[FS] G. Fix and G. Strang, Fourier Analysis of the finite element method in Ritz-Galerkin theory, Studies in Appl. Math. **48** (1969), 265-273.

[S] G. Strang, The finite element method and approximation theory, in : Numerical solution of partial differential equations, SYNSPADE 1970, Univ. of Maryland, College Park, B. Hubbard, ed. 1970, pp. 547-583.

[SF] G. Strang and G. Fix, A Fourier analysis of the finite element variational method, C.I.M.E.II, Ciclo Erice, 1971, Constructive Aspects of Functional Analysis, G. Geymonat, ed. 1973, 793-840.

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER #2743	2. GOVT ACCESSION NO. A147340	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) A Counterexample to a Result of Strang and Fix Concerning Controlled Approximation		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Rong-Qing Jia		8. CONTRACT OR GRANT NUMBER(s) DAAG29-80-C-0041 MCS-8210950
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin 610 Walnut Street Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 3 - Numerical Analysis and Scientific Computing
11. CONTROLLING OFFICE NAME AND ADDRESS (See Item 18 below)		12. REPORT DATE September, 1984
		13. NUMBER OF PAGES 13
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

U.S. Army Research Office
P.O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, DC 20550

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Controlled Approximation, Approximation Order, Box Splines, Finite Element Analysis, Fourier Series

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A result of Strang and Fix states that if the order of controlled approximation from a collection of locally supported elements is k, then there is a finite linear combination Ω of those elements and their translates such that any polynomial of degree less than k can be reproduced by Ω and its translates. This paper gives a counterexample to their result.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

END

FILMED

12-84

DTIC